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Spinorial cohomology and maximally supersymmetric theories

Martin Cederwall, Bengt E.W. Nilsson and Dimitrios Tsimpis

Department of Theoretical Physics Göteborg University and
Chalmers University of Technology
SE-412 96 Göteborg, Sweden
E-mail: martin.cederwall@fy.chalmers.se, bengt.nilsson@fy.chalmers.se, tsimpis@fy.chalmers.se

Abstract: Fields in supersymmetric gauge theories may be seen as elements in a spinorial cohomology. We elaborate on this subject, specialising to maximally supersymmetric theories, where the superspace Bianchi identities, after suitable conventional constraints are imposed, put the theories on shell. In these cases, the spinorial cohomologies describe in a unified manner gauge transformations, fields and possible deformations of the models, e.g. string-related corrections in an α' expansion. Explicit cohomologies are calculated for super-Yang-Mills theory in $D = 10$, for the $\mathcal{N} = (2,0)$ tensor multiplet in $D = 6$ and for supergravity in $D = 11$, in the latter case from the point of view of both the super-vielbein and the super-3-form potential. The techniques may shed light on some questions concerning the α'-corrected effective theories, and result in better understanding of the rôle of the 3-form in $D = 11$ supergravity.

Keywords: Field Theories in Higher Dimensions, Supergravity Models, Superspaces.

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Contents

1. Introduction

The purpose of this paper is to investigate an interesting structure of supersymmetric field theories, recently found when deriving conditions on interactions in maximally supersymmetric Yang-Mills theory [1, 2, 3]. It was observed that component fields and gauge transformations, as well as physically distinguishable deformations of the model, are represented as elements in cohomology classes under a certain fermionic exterior derivative. Since the model under consideration, super-Yang-Mills theory in $D=10$, is a maximally supersymmetric model, known to possess no manifestly supersymmetric off-shell formulation, the deformations are represented (at least at the linearised level) as a current supermultiplet. In addition to promoting a better understanding of the mechanisms at hand for such theories, the concept turned out to be quite efficient in understanding field redefinitions relating physically equivalent deformations.

The structure we are dealing with is a sequence of representations of the global symmetry, which is the Lorentz group L (together with the R-symmetry group R if there is one — most of the cases we deal with in the present paper have trivial R-symmetry). The examples we will treat are 10-dimensional super-Yang-Mills theory [4], the $\mathcal{N}=(2,0)$ tensor multiplet in $D=6$ and 11-dimensional supergravity [5, 6], the latter one both from the perspective of the vielbein and from that of the three-form tensor. The treatment of dimensional reductions of these theories will follow from dimensional reduction of the complexes. Another case, which we expect to show similar properties, is type IIB supergravity in $D=10$, which we will not treat in this paper. $\mathcal{N}=(1,0)$ super-Yang-Mills in $D=6$ will also be analysed, as a contrasting example of a theory possessing a supersymmetric off-shell formulation.

The basic idea is that the theories we consider are gauge theories, and that, in a superspace formulation, where all potentials and field strengths are forms on superspace, all components except the purely spinorial ones are redundant. Since all physical fields are contained in the objects carrying spinorial form indices only, it is interesting to examine the structure arising from these. Our complexes are of the form

$$r_0 \xrightarrow{\Delta_0} r_1 \xrightarrow{\Delta_1} r_2 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{n-1}} r_n \xrightarrow{\Delta_n} \ldots\,, \qquad (1.1)$$

where r_p, for some $p \geq 0$, is the representation carried by a gauge transformation, r_{p+1} that of a potential and r_{p+2} that of a field strength. We will refer to the representations r_n as n-forms, a notation not to be confused with that of a tensor antisymmetric in vector indices. The exact definitions are given, both for gauge theory and supergravity, in the following sections, where it will also be clear why Δ is a nilpotent operator. The rôle of r_{p+3} is as a Bianchi identity.

A supersymmetric gauge theory or a supergravity theory, when formulated on superspace, has to fulfill a number of Bianchi identities. These are of course trivial as long as field strengths are defined from potentials, but become non-trivial when conventional constraints are imposed on the field strengths. Then one can use the Bianchi identities as integrability conditions and work only at the level of field strengths. For the maximally supersymmetric theories we are dealing with, this procedure enforces equations of motion for the component fields, which however depend on the choice of the fermionic components of the field strengths, which in a certain sense, explained in the following, play the rôles of current multiplet superfields. Since the structure we are presenting only deals with the totally fermionic forms, we ignore all Bianchi identities except the spinorial one. We are thus not performing a complete analysis of the theories, for which the reader should consult other references [1, 2] and [7]–[10]. The analysis of this paper demonstrates, among other things, the *possibility* of deforming the theories in a non-trivial manner by turning on field strengths. That such a procedure is consistent follows from a full analysis of the Bianchi identities, which will also yield the exact form of the deformations. This has been done for $D = 10$ super-Yang-Mills [1, 2], while for $D = 11$ supergravity a partial analysis has been performed [8] and a more complete one is envisaged [9].

2. Spinorial complexes and cohomology

The structure of the complex is

$$r_0 \xrightarrow{\Delta_0} r_1 \xrightarrow{\Delta_1} r_2 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{n-1}} r_n \xrightarrow{\Delta_n} \ldots\,, \qquad (2.1)$$

where the representations r_n at each n denotes a superfield in the representation r_n of the Lorentz group. When we describe a gauge theory, r_n consists of totally symmetric and Γ-traceless tensors in n spinor indices. The fermionic exterior derivative is a projection on the representations r_n of a (symmetrised) spinorial covariant derivative. The general interpretation is that r_0 contains gauge transformations, r_1 contains fields and r_2 deformations of the theory. For a theory of a rank-$(p+1)$ tensor potential, such as the three-form in $D = 11$ supergravity, r_p contains gauge transformations, r_{p+1} fields and r_{p+2} deformations. The representations in a complex associated with the vielbein of a supergravity theory are slightly different. Then r_0 is a vector, corresponding to reparametrisation gauge transformations, r_1 is a vector-spinor, and so on.

It is convenient to decompose the representations in component fields sitting at levels ℓ, i.e., multiplying θ^ℓ in the superfields. We write those as $r_n^\ell \equiv \wedge^\ell S \otimes r_n$, where S is the representation of the spinorial derivative (i.e., the conjugate representation to that of θ). Appendix B gives, as an example, all antisymmetric products of chiral spinors in $D = 10$ [22].

We will argue from very general arguments that the interesting content at each n actually is the cohomology with respect to the exterior derivative Δ, $\mathcal{H}^n = \mathrm{Ker}\Delta_n / \mathrm{Im}\Delta_{n-1}$. We shall give a couple of examples of this for maximally supersymmetric theories, and also compare to the situation in theories with lower supersymmetry.

We use the Dynkin labels of highest weights to denote irreducible representations of $L \times R$. The $D = 10$ super-Yang-Mills complex is:

$$(00000) \xrightarrow{\Delta_0} (00010) \xrightarrow{\Delta_1} (00020) \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{n-1}} (000n0) \xrightarrow{\Delta_n} \ldots . \tag{2.2}$$

The complex for the $\mathcal{N} = (2, 0)$ tensor multiplet in $D = 6$, with $R = \mathrm{Sp}(4)$, is

$$\begin{tikzcd}
(000)(00) \arrow[r] & (100)(10) \arrow[r] \arrow[dr] & (200)(20) \arrow[r] \arrow[dr] & (300)(30) \arrow[r] \arrow[dr] & (400)(40) \ldots \\
 & & (010)(01) \arrow[r] & (110)(11) \arrow[r] \arrow[dr] & (210)(21) \ldots \\
 & & & & (020)(02) \ldots
\end{tikzcd} \tag{2.3}$$

The corresponding complex in $D = 11$, which we will apply to the 3-form present in $D = 11$ supergravity, is

$$\begin{tikzcd}
(00000) \arrow[r] & (00001) \arrow[r] \arrow[dr] & (00002) \arrow[r] \arrow[dr] & (00003) \arrow[r] \arrow[dr] & (00004) \arrow[r] \arrow[dr] & (00005) \ldots \\
 & & (01000) \arrow[r] & (01001) \arrow[r] \arrow[dr] & (01002) \arrow[r] \arrow[dr] & (01003) \ldots \\
 & & & & (02000) \arrow[r] & (02001) \ldots
\end{tikzcd} \tag{2.4}$$

The $D = 11$ supergravity complex, which is obtained from the last example by adding the vector highest weight (10000), is:

$$\begin{tikzcd}
(10000) \arrow[r] & (10001) \arrow[r] \arrow[dr] & (10002) \arrow[r] \arrow[dr] & (10003) \arrow[r] \arrow[dr] & (10004) \ldots \\
 & & (11000) \arrow[r] & (11001) \arrow[r] \arrow[dr] & (11002) \ldots \\
 & & & & (12000) \ldots
\end{tikzcd} \tag{2.5}$$

In contrast to the one for $D = 10$ SYM, the $D = 11$ complexes contain reducible representations for $n \geq 2$, simply because the symmetric bi-spinors, apart from the vector, decompose into an 2-form, (01000), and a 5-form, (00002). The situation in $D = 6$ is similar; here the symmetric bispinors contain a vector which is an Sp(4) singlet, (010)(00), a vector in **5** of Sp(4), (010)(01), and a self-dual 3-form in **10** of Sp(4), (200)(20).

Here we have not bothered to name the operators taking us between irreducible representations, only indicated with arrows which paths are possible.

Let us describe in more detail how the complexes work, with the super-Yang-Mills theory as an example. The gauge potentials are A_α and A_a. However, the spinor potential

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already contains a vector (of correct dimension) at the θ level, and this is the reason why a conventional constraint is needed in order to have *one* vector potential. This constraint is

$$\Gamma_a^{\alpha\beta} F_{\alpha\beta} = 0\,, \tag{2.6}$$

which implies that

$$A_a = -\frac{1}{16}\left(D\Gamma_a A - A\Gamma_a A\right)\,. \tag{2.7}$$

The rest of $F_{\alpha\beta}$,

$$F_{\alpha\beta} = \frac{1}{5!}\Gamma_{\alpha\beta}^{a_1\ldots a_5} J_{a_1\ldots a_5}\,, \tag{2.8}$$

which lies in (00020), does not contain A_a. We also note [11, 12, 1] that part of the dimension-1/32 Bianchi identity states the vanishing of the (00030) component of $D_\alpha F_{\beta\gamma}$. These observations make it natural to consider, not the sequence of completely symmetric representations in spinor indices, but a restriction of it, namely the sequence of Spin(1,9) representations in eq. (2.2). The representation $r_n \equiv (000n0)$ is the part of the totally symmetric product of n chiral spinors that has vanishing "Γ-trace", and may be represented tensorially as $C_{\alpha_1\ldots\alpha_n} = C_{(\alpha_1\ldots\alpha_n)}$, $\Gamma_a{}^{\alpha_1\alpha_2} C_{\alpha_1\alpha_2\alpha_3\ldots\alpha_n} = 0$. For $n = 2$, C is an anti-selfdual five-form, for $n = 3$ a Γ-traceless anti-selfdual five-form spinor, etc.

The operator Δ_n: $r_n \longrightarrow r_{n+1}$ can schematically be written as $\Delta_n C_n = \Pi(r_{n+1}) D C_n$, where D is the exterior covariant derivative $D = d\theta^\alpha D_\alpha$ and $\Pi(r_n)$ is the algebraic projection from $\otimes_s^n(00010)$ to $(000n0)$. It is straightforward to write an explicit tensorial form for Δ by subtracting Γ-traces from DC, but it will not be used here. It is also straightforward to show that, for an abelian gauge group and standard flat superspace, the sequence (2.2) forms a complex, i.e., that $\Delta^2 = 0$. This follows simply from the fact that while $\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c$, the torsion only has a component $2\Gamma_{\alpha\beta}{}^c$ which is projected out by $\Pi(r_n)$. The anticommutator of two covariant spinor derivatives is in general

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c - T_{\alpha\beta}{}^\gamma D_\gamma + F_{\alpha\beta}\cdot + R_{\alpha\beta}{}^{\cdot}\,, \tag{2.9}$$

and Δ will not be nilpotent in arbitrary curved backgrounds or non-abelian theories. In these cases, we must consider the complex for an undeformed super-Yang-Mills or supergravity theory, and consider *infinitesimal* deformations as elements of the cohomologies. That eq. (2.9) in the undeformed theories yield $\Delta^2 = 0$ is seen as follows. For the Yang-Mills case, $F_{\alpha\beta} = 0$ in the undeformed theory. For $D = 11$ supergravity the argument for $\Delta^2 = 0$ in the undeformed theory is slightly more complicated. One has to remember that $T_{a\beta}{}^\gamma$ is non-zero (it contains the 4-form tensor field strength H). The torsion Bianchi identities at dimension 1 give

$$\begin{aligned} R_{(\alpha\beta\gamma)}{}^\delta &= 6\Gamma_{(\alpha\beta}{}^e T_{|e|\gamma)}{}^\delta\,, \\ R_{\alpha\beta c}{}^d &= -4T_{c(\alpha}{}^\gamma \Gamma_{\beta)\gamma}{}^d\,. \end{aligned} \tag{2.10}$$

Letting two consecutive spinorial derivatives act on an element $C_{\gamma_1\ldots\gamma_n}$ in the sequence (2.4) or $C_{\gamma_1\ldots\gamma_n}{}^c$ in (2.5) gives additional curvature terms according to eq. (2.9). Inserting the expressions for the dimension-1 curvature of eq. (2.10) implies that the resulting expressions can be written as

$$
\begin{aligned}
D_{(\alpha}D_{\beta}C_{\gamma_1\ldots\gamma_n)} &= \Gamma_{(\alpha\beta}{}^{d}\phi_{|d|\gamma_1\ldots\gamma_n)}\,,\\
D_{(\alpha}D_{\beta}C_{\gamma_1\ldots\gamma_n)}{}^{c} &= \Gamma_{(\alpha\beta}{}^{d}\psi_{|d|\gamma_1\ldots\gamma_n)}{}^{c} + \Gamma^{c}{}_{(\alpha}{}^{\delta}\chi_{|\delta|\beta\gamma_1\ldots\gamma_n)}\,.
\end{aligned}
\tag{2.11}
$$

Each of these terms vanish under the projection on the irreducible representations constituting the complexes — while the representations r_n are "Γ-traceless", they contain pure Γ-traces only.

We would now like to calculate the cohomology of the complex associated with $D = 10$ super-Yang-Mills. This can be done by considering the decomposition into irreducible representations of the representation sitting at level ℓ in r_n, $r_n^\ell \equiv \wedge^\ell S \otimes r_n$. This is easily done, e.g. with the help of the program LiE [15]. One then follows each of the irreducible representations at a given dimension through the subcomplex

$$
r_0^\ell \to r_1^{\ell-1} \to r_2^{\ell-2} \to \ldots \to r_{\ell-1}^1 \to r_\ell\,. \tag{2.12}
$$

Let us illustrate the calculation by examining the field content. We then look into the spinor potential of dimension 1/2, which contains all fields in the vector multiplet, so we should examine the first cohomology. The vector (dimension 1) sits at $\ell = 1/2$ and the spinor (dimension 3/2) at $\ell = 1$. The subcomplexes under consideration are $r_0^2 \to r_1^1 \to r_2$ and $r_0^3 \to r_1^2 \to r_2^1 \to r_3$. Checking the multiplicities of the relevant representations, (10000) and (00001), in these, we obtain the sequences $0 \to 1 \to 0$ and $0 \to 1 \to 0 \to 0$. The components of the cohomology in these representations and dimensions clearly contain the physical fields. This can be understood in a traditional framework as removing degrees of freedom in a superfield gauge transformation (removing the image from the left) and imposing the vanishing of the field strength $F_{\alpha\beta}$ (removing the complement of the kernel from the right). Analogous considerations tell us that the second cohomology contains a spinor of dimension 5/2 and a vector of dimension 3. These are interpreted as belonging to a current supermultiplet, i.e., fields entering the right hand sides of the equations of motion. This goes well together with the observation that modifications of the theory are introduced by deforming the constraint $F_{\alpha\beta} = 0$ [11, 13, 14, 1, 2]. The relevance of the cohomology is explained by the facts that deformations introduced by relaxing $F_{\alpha\beta} = 0$ have to fulfill the Bianchi identity (removing the complement of the kernel from the right), and that relevant deformations are counted modulo field redefinitions (removing the image from the left). See also the following section for a fuller discussion.

A complete calculation of the cohomology requires that one considers all irreducible representations occurring at arbitrary levels. This quickly becomes untractable to do by hand. Unfortunately, there is also another complication, that makes it impossible to derive the cohomologies unambiguously from multiplicities only without making further assumptions. This can be exemplified by looking at some other representation; let us take the 3-form at dimension 3. The subcomplex $r_0^6 \to r_1^5 \to r_2^4 \to r_3^3 \to r_4^2 \to r_5^1 \to r_6$ yields the multiplicities $0 \to 0 \to 1 \to 1 \to 0 \to 0 \to 0$. We do not expect any non-zero cohomology, since there is no equation of motion in this representation. Yet, the sequence of multiplicities offers two possibilities: either there is one 3-form in the first cohomology and one in the second, or there is none at all. Using tensorial methods, it is easy to show that the 3-form in r_2^4 has an image in r_3^3, so the cohomology vanishes. In the present paper, we will make the assumption of "maximal propagation" of irreducible representations through the subcomplexes, meaning that representations have images or belong to images under Δ if possible. This assumption is enough to determine the super-Yang–Mills cohomology completely, at least for $n \leq 5$. The result, derived already in ref. [1], is presented in table 1. We expect higher cohomologies to vanish. The method for calculating cohomologies, under the assumption of maximal propagation, is by using the code of appendix A with the program LiE [15]. As we will see, there are cases in $D = 11$ where even the assumptions made so far leave an ambiguity. The reason that we choose to make educated guesses rather than turn to tensor calculations is that the number of irreducible representations is so large that such a treatment becomes virtually impossible.

We now turn to the cohomology of the complex (2.5) associated with the super-vielbein of $D = 11$ supergravity. All fields in the supergravity multiplet are contained in the dimension-$(-1/2)$ vielbein $E_\alpha{}^a$ (actually in the Γ-traceless part (10001)), which plays an analogous rôle to that of A_α in super-Yang–Mills theory. All other components should be related to this one by conventional constraints. When one considers the corresponding field strength, the dimension-0 torsion component $T_{\alpha\beta}{}^a$, it is known [16, 17, 18, 8] that the only components surviving after imposing conventional constraints are the usual constant Γ-matrix term, and two fields $X_{a_1a_2}{}^a$ in (11000) and $X_{a_1\ldots a_5}{}^a$ in (10002) entering the torsion as

$$T_{\alpha\beta}{}^a = 2\left(\Gamma^a_{\alpha\beta} + \frac{1}{2}\Gamma^{a_1a_2}_{\alpha\beta}X_{a_1a_2}{}^a + \frac{1}{5!}\Gamma^{a_1\ldots a_5}_{\alpha\beta}X_{a_1\ldots a_5}{}^a\right) . \tag{2.13}$$

The part of the torsion Bianchi identity with purely spinorial form indices is

$$D_{(\alpha}T_{\beta\gamma)}{}^a|_{(11001)\oplus(10003)} = 0 \tag{2.14}$$

(analagously to eq. (2.9), there are higher order torsion terms that should be added to this equation, which do not contribute to the relevant representations for infinitesimal deformations). We are naturally led to consider the sequence of representations already stated in eq. (2.5).

The cohomology is given in table 3, but we would like to illustrate part of the calculation. The representations in the zeroth and first cohomologies are calculated the same way as for super-Yang–Mills. When we come to the second cohomology, representing the current supermultiplet contained in the torsion components (11000) and (10002) at dimension 0 [8], there is a complication illustrated by the following example. Take the spinor (00001) at dimension $\frac{3}{2}$. It will occur in the equation of motion for the gravitino, if it is contained in the $n = 2$ cohomology (that this is the case, and that it does not affect the Weyl curvature at dimension $\frac{3}{2}$, was actually shown in ref. [8]). We now only write the multiplicity of the representation in each r_n^ℓ, and get the sequence $1 \to 3 \to 3 \to 0 \to 0 \to 0$. This sequence offers two distinct possibilities even under the assumption of maximal propagation: either the representation in r_0 has an image among the three in r_1, in which case there is a cohomology in r_2, or all three representations in r_2 have images in r_3, in which case there is a cohomology in r_0. In this specific case, tensorial methods have already been used [8] that

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
dim $=0$	(00000)				
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

Table 1: The cohomology of the $D = 10$ SYM complex.

show that the first of these possibilities is true, so that the second cohomology contains a spinor in the equations of motion. The rôle of the code given in appendix A is that it makes use of the assumption of maximal propagation, and in cases like the one just related, gives candidate cohomologies in all possible cases. For the cohomologies associated with the $D = 11$ super-vielbein and super-3-form, such ambiguities exist when one goes higher in dimension than those of the fields, and the results of tables 3 and 4 consist partially of educated guesses concerning which alternatives are the correct ones. We are led in part by expectations concerning the content of the current supermultiplet and in part by the resulting symmetry of the tables — all three cohomologies in 10 and 11 dimensions seem to have an inherent symmetry under reflection in one point (in the $D = 10$ case accompanied by a $\mathbb{Z}_2$ automorphism exchanging the spinor representations of Spin(1,9)).

To summarise the calculation of the cohomologies, the non-vanishing cohomology associated to $D = 10$ super-Yang–Mills theory is

$$\begin{aligned}
\mathcal{H}^0 &= (00000)_0 \,, \\
\mathcal{H}^1 &= (10000)_1 \oplus (00001)_{\frac{3}{2}} \,, \\
\mathcal{H}^2 &= (00010)_{\frac{5}{2}} \oplus (10000)_3 \,, \\
\mathcal{H}^3 &= (00000)_4
\end{aligned} \tag{2.15}$$

(the dimensions are given as subscripts), or represented graphically in table 1, divided in different n and ℓ (the dimension is $(n + \ell)/2$)

The cohomology for the $\mathcal{N} = (2, 0)$ tensor multiplet in $D = 6$ and the $D = 11$ cohomologies for the super-vielbein and the tensor are given in tables 2, 3 and 4:

3. The meaning of the cohomologies

The maximally supersymmetric theories we are considering have the property that imposing the vanishing of certain lowest-dimensional field strengths, (J and the X tensors) implies the equations of motion. This is related to the non-vanishing second cohomology. The second cohomology contains a current supermultiplet, that enters the field equations

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim=0$	(000)(00)				
$\frac{1}{2}$	•	•			
1	•	(010)(00)	•		
$\frac{3}{2}$	•	•	•	•	
2	•	•	(000)(01) (101)(00)	•	•
$\frac{5}{2}$	•	•	(100)(10)	•	•
3	•	•	•	(002)(00)	•
$\frac{7}{2}$	•	•	•	(001)(10)	•
4	•	•	•	(000)(01)	•
$\frac{9}{2}$	•	•	•	•	•

Table 2: The cohomology of the $D=6$, $\mathcal{N}=(2,0)$ complex.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\dim=-1$	(10000)						
$-\frac{1}{2}$	(00001)	•					
0	•	(20000)	•				
$\frac{1}{2}$	•	(00001) (10001)	•	•			
1	•	(00010) (10000)	•	•	•		
$\frac{3}{2}$	•	•	(00001) (10001)	•	•	•	
2	•	•	(00000)(00002) (00100)(01000) (10000)(20000)	•	•	•	•
$\frac{5}{2}$	•	•	•	•	•	•	•
3	•	•	•	(00000)(00002) (00100)(01000) (10000)(20000)	•	•	•
$\frac{7}{2}$	•	•	•	(00001) (10001)	•	•	•
4	•	•	•	•	(00010) (10000)	•	•
$\frac{9}{2}$	•	•	•	•	(00001) (10001)	•	•
5	•	•	•	•	(20000)	•	•
$\frac{11}{2}$	•	•	•	•	•	(00001)	•
6	•	•	•	•	•	(10000)	•
$\frac{13}{2}$	•	•	•	•	•	•	•

Table 3: The $D=11$ supergravity cohomology, with respect to the super-vielbein.

implied by the whole set of Bianchi identities (that we do not consider in this paper). If the constraints for the field strengths are changed, the equations of motion change.

To understand this, suppose for a moment that those cohomologies vanished. Then, whatever constraint was put upon the field strengths (consistent with the Bianchi identities) could be expressed as the image of an exterior derivative acting on a 1-form (a potential). Such a field strength could be removed by a field redefinition, and the system would be equivalent to one where those field strength components were set to zero. One might ask

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	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$\dim=-3$	(00000)								
$-\frac{5}{2}$	•	•							
-2	•	(10000)	•						
$-\frac{3}{2}$	•	•	•	•					
-1	•	•	(01000) (10000)	•	•				
$-\frac{1}{2}$	•	•	(00001)	•	•	•			
0	•	•	•	(00000) (00100) (20000)	•	•	•		
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•	•	
1	•	•	•	•	•	•	•	•	•
$\frac{3}{2}$	•	•	•	•	(00001) (10001)	•	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•	•
$\frac{5}{2}$	•	•	•	•	•	(00001)	•	•	•
3	•	•	•	•	•	(01000) (10000)	•	•	•
$\frac{7}{2}$	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•	•
$\frac{9}{2}$	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)	•
$\frac{11}{2}$	•	•	•	•	•	•	•	•	•

Table 4: The $D=11$ supergravity cohomology, with respect to the super-3-form.

whether it is not inconsistent to treat the field strength as belonging to the cohomology, since it is supposed to be derived from a potential, and in that sense cohomologically trivial. In fact, what the remaining Bianchi identities do is to resolve the cohomology in the sense that the field strength indeed comes from a potential. This is achieved by the equations of motion. Take e.g. the equation of motion for the spinor λ in super-Yang-Mills. In the undeformed case it reads $\not{D}\lambda=0$. When deformations are turned on, it will read $\not{D}\lambda=\mu$, and if μ is derived from a cohomologically trivial J, μ will be of the form $\mu=\not{D}\nu$. Then the equation of motion for λ is modified in a trivial way, removable by $\lambda-\nu\rightarrow\lambda$. Only if J is cohomologically non-trivial do the equations of motion receive significant modifications. Once the equations of motion are taken into account, they state exactly the integrability of the field strength to a potential (in the example that μ is $\not{D}$ on something). So, the cohomology is resolved by the field equations, but if it was trivial from the beginning, nothing would have changed.

It is instructive to compare with a non-maximally supersymmetric gauge theory known to possess an off-shell superfield formulation. Take the $\mathcal{N}=(1,0)$ super-Yang–Mills theory in $D=6$ [20]. This theory has an off-shell formulation in terms of the vector, the spinor and a triplet of auxiliary scalars of dimension 2. The complex is [1, 3]

$$(000)(0)\xrightarrow{\Delta_0}(100)(1)\xrightarrow{\Delta_1}(200)(2)\xrightarrow{\Delta_2}\ldots\xrightarrow{\Delta_{n-1}}(n00)(n)\xrightarrow{\Delta_n}\ldots \tag{3.1}$$

Indeed, the only non-vanishing cohomologies are

$$\begin{aligned}\mathcal{H}^0 &= (000)(0)_0 \,,\\ \mathcal{H}^1 &= (010)(0)_1 \oplus (001)(1)_{\frac{3}{2}} \oplus (000)(2)_2 \,,\end{aligned} \tag{3.2}$$

where the representations are given as standard Dynkin labels for Spin(1,5)×SU(2) (the second factor being the R-symmetry group). The second cohomology is trivial, which also is expected — setting $F_{\alpha\beta}$ to zero does not put the theory on-shell, and the value of $F_{\alpha\beta}$ does not contain any information about interactions — it can be set to zero by a field redefinition.

It should be noted that even if this picture is quite clear for both $D = 10$ super-Yang-Mills and $D = 11$ supergravity, no explicit such understanding has been achieved for a formulation of $D = 11$ supergravity based on the super-3-form. We comment more on this below.

There is a striking resemblance [24] between spinorial cohomology as it is constructed in this paper and the BRST cohomology in Berkovits' covariant formulation of superstrings using pure spinors [25]. It seems as the choice of representations building our complexes amounts to the same information encoded in the contraction of the spinor derivative into a pure spinor. Further investigation of this similarity, as well as the relation to the pure spinors of ref. [12] should be pursued.

Reading the tables of cohomologies, a number of observations can be made. Starting with the super-Yang-Mills case, table 1, the interpretation is clear. The gauge transformations (zeroth cohomology) and fields (first cohomology) are the usual ones, and the second cohomology in the field strength contains exactly the representations fitting into the right hand sides of equations of motion for the spinor and vector. This case has been worked out in full detail [1]. The only element of the cohomology that has not yet been explained is the scalar of dimension 4 in the third cohomology. It has the correct dimension for a lagrangian density, and we suspect that it might be related to an action principle containing the deformation of the theory (not the ordinary kinetic terms).

In table 2, the picture is similar. The first cohomology contains the gauge transformations, the second one the five scalars, the antisymmetric tensor and the two spinors, while the third cohomology carries the representations of the currents: an (anti-)selfdual tensor giving the selfduality of the 3-form field strength, spinors of the opposite chirality and five scalars, all at appropriate dimensions.

Turning to table 3, and the $D = 11$ supergravity, the zeroth cohomology clearly represents the bosonic and fermionic reparametrisations. In the first cohomology, we find the fields: at dimension 0 the (linearised) metric, at dimension 1/2 the gravitino in (10001) and at dimension 1 the 4-form field strength. In addition, there is a spinor at dimension 1/2 and a vector at dimension 1. The experience from solving superspace Bianchi identities for this system [7, 8, 19] tells us that these should be identified with the spinor and vector components of the Weyl connection 1-form.[1] The second cohomology contains the

[1] Although we use a superspace that does not have Weyl scalings as part of its structure group, we refer to these as Weyl connections, since they appear in the torsion in exactly the places where they could be absorbed in a Weyl connection by a conventional constraint.

representations for the "usual" equations of motion: (10001) $\oplus$ (00001) for the gravitino, (00000) $\oplus$ (20000) for the metric and (00100) for the 3-form potential. Since the latter one, due to gauge invariance, appears in the geometry only through its field strength, one also finds its Bianchi identity in (00002) here. In addition there are the representations (01000) $\oplus$ (10000) at dimension 2. One of the controversies over the possible deformations of the eleven-dimensional supergravity theory concerns the rôle of the Weyl connections. It has been conjectured that the corresponding curvatures vanish, so that the they are integrable to a scalar field, the Weyl compensator, that can be removed by a conventional constraint [7, 8], but also that they should play a significant rôle in a deformed theory [19]. In ref. [8], we found evidence for the first of these alternatives in the fact that the spinor at dimension 3/2 affects only the gravitino equation of motion, not the Weyl curvature.[2] This of course does not follow from the present listing of representations, but requires exact solution of the Bianchi identities. It was also found that the dimension-1 part of the Weyl curvature, $G_{\alpha\beta}$, vanished, which looks natural in the absence of any such cohomology. In the undeformed theory, all components of the Weyl curvature vanish, but this has not been completely shown in a theory with deformation by the X-tensors. We see that there *a priori* is room in the second cohomology for a modification to the dimension-2 Weyl curvature G_{ab} in (01000), but a treatment of the full set of Bianchi identities [9] has to be awaited to determine whether it is actually there. From the present analysis, it is also not ruled out that there might be corrections to the Bianchi identity for the 4-form field strength. Concerning the vector at dimension 2 we do not have any natural interpretation, and find it plausible that it goes away when the full set of Bianchi identities is considered.

Table 4, containing the cohomologies for an antisymmetric tensor in $D = 11$ (the dimensions are adapted to a 3-form potential), gives some new information. We are used to the fact that the tensor field arises in a superspace formulation from super-geometry only, so that the 4-form field strength sits inside the torsion at dimension 1, and that the closed super-4-form is constructed out of geometrical data. The present analysis indicates that it is possible to turn this around, and see the geometry as arising from the dynamics of a super-3-form potential. It was observed in ref. [8] that the super-4-form, once the deformations are turned on, is forced to have non-vanishing components at negative dimensions. Here we will argue that these actually can encode the deformations, and that this implies integrability conditions on the X-tensors, not readily visible from the geometrical analysis but presumably hidden therein.

The appearance of components at negative dimension immediately gives rise to questions concerning propagation of branes and BPS conditions in deformed backgrounds. Components of negative dimension in the brane Wess-Zumino terms have no counterpart in the kinetic term, which complicates the issue of κ-symmetry [21]. Our opinion is that the nature of κ-symmetry has to be changed in a deformed theory, so that the local transformation parameter κ, seen as a vector field on superspace, will have a non-vanishing Lorentz vector part. In the embedding formalism [17], this would manifest itself as a deformation of the embedding constraint.

[2]In that analysis, only the X-tensor in (10002), not the one in (11000), was included for simplicity.

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In the second cohomology, representing gauge transformations with parameter $\Lambda_{\alpha\beta}$ in (01000) $\oplus$ (00002), we find the 2-form gauge transformation for the tensor field as well as bosonic and fermionic reparametrisations. In the third cohomology, all the supergravity fields are contained in the potential $C_{\alpha\beta\gamma}$ in (01001) $\oplus$ (00003): at dimension 0 the 3-form potential and the graviton, and at dimension 1/2 the gravitino. In addition, there is a scalar at dimension 0 and a spinor at dimension 1/2. A tentative interpretation is that they represent the Weyl compensator and the spinorial Weyl connection. The fourth cohomology, sitting in the superfield $H_{\alpha\beta\gamma\delta}$ in (02000) $\oplus$ (01002) $\oplus$ (00004), contains equations of motion for the gravitino, for the metric and for the 3-form, and no extra representations. Since now the 3-form appears directly, and not via its field strength, there is no need for its Bianchi identity in the fourth cohomology. We observe that there is room neither for the vectorial Weyl connection not for its field strength. There is still a possibility for an equation of motion for the spinor to mix in with the spinor part of the gravitino equation of motion, but in the light of what the geometrical analysis shows (see the previous paragraph) this seems very unlikely.

It would be very interesting to analyse the $D = 11$ supergravity from the point of view of the 3-form instead of the super-geometry. How this is done at a linearised level is obvious, but how a full treatment should be performed without invoking geometry is completely unclear, although the present analysis indicates that it might be possible. Such an approach might offer new perspectives on M-theory, and we hope to be able to investigate it in the future. One question that can be addressed in a traditional treatment is whether or not the requirement that there exist a closed superspace 4-form puts stronger restrictions on the system than does a purely geometric analysis. We are primarily aiming at the controversy about the fields connected to Weyl scalings. Our guess is that the two approaches should be equivalent, and that the Bianchi identities of higher dimensions not considered in this paper enforce triviality of the Weyl bundle, as shown in ref. [7] for the undeformed theory and conjectured in ref. [8] for the deformed theory. If this statement is true, the 3-form approach may actually offer some advantages over the geometric picture, since it encodes the two-step integrability of the X-tensors to the three irreducible representations in $H_{\alpha\beta\gamma\delta}$ (this integrability goes in another direction than the complex for the super-vielbein), something that may be very useful when one wants to express these in terms of physical fields in the supergravity multiplet to get explicit α'-corrections from supersymmetry. We believe that although the attempts made so far [8] have only been partially successful, due to the technical complexity of the calculations, a heavier use of computer techniques [15, 23] will render the investigations [9] tractable.

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A. The LiE code

The code used with the program LiE to calculate cohomologies in $D = 10$ is:

```
##### definitions for D=10 SYM #####
maxobjects 1000000
setdefault D5
rank=5
r(int n)=1X[0,0,0,n,0]
s=[0,0,0,1,0]
##### calculate content of superfields #####
r(int m,n)=
{if n==0 then r(m) else
     if n<0 then poly_null(rank) else
          if m<0 then poly_null(rank) else
               tensor(r(m),alt_tensor(n,s))
          fi;
     fi;
fi;
}
##### set negative multiplicities to zero #####
pos_pol(pol p)=
{loc q=p;
for i=1 to length(p) do
     if coef(p,i)<0 then q=q-p[i];
     fi;
od;
q}
##### subtract multiplicities from the left #####
left(int m,n)=
{loc t=poly_null(rank);
if m-1<0 then
     r(m,n);
else
     t=r(0,m+n);
     for k=1 to m do
          t=pos_pol(r(k,m+n-k)-t);
     od;
     t;
fi
}
##### subtract multiplicities from the right #####
right(int m,n)=
{loc t=poly_null(rank);
if n-1<0 then
     r(m,n);
```

```
else
     t=r(m+n,0);
     for k=1 to n do
          t=pos_pol(r(m+n-k,k)-t);
     od;
     t;
fi
}
##### calculate candidate cohomologies #####
h(int m,n)=pos_pol(r(m,n)-right(m+1,n-1)-left(m-1,n+1))
```

For the other three cases, the first part of the code (definitions) is replaced by

```
##### definitions for D=6 tensor #####
maxobjects 1000000
setdefault A3C2
rank=5
s=1X[1,0,0,1,0]
##### build up reducible r(n) iteratively #####
r(int n)=
{loc q=poly_null(rank);
if n%2==0 then
     loc k=n/2;
     for i=0 to k do
          q=q+1X[n-2*i,i,0,n-2*i,i];
     od;
fi;
if n%2==1 then
     loc k=(n-1)/2;
     for i=0 to n/2 do
          q=q+1X[n-2*i,i,0,n-2*i,i];
     od;
fi;
q;
}
```

by

```
##### definitions for D=11 SG #####
maxobjects 1000000
setdefault B5
rank=5
s=1X[0,0,0,0,1]
##### build up reducible r(n) iteratively #####
r(int n)=
```

```
{loc q=poly_null(rank);
if n%2==0 then
     loc k=n/2;
     for i=0 to k do
          q=q+1X[1,i,0,0,n-2*i];
     od;
fi;
if n%2==1 then
     loc k=(n-1)/2;
     for i=0 to n/2 do
          q=q+1X[1,i,0,0,n-2*i];
     od;
fi;
q;
}
```

and by

```
##### definitions for D=11 tensor #####
maxobjects 1000000
setdefault B5
rank=5
s=1X[0,0,0,0,1]
##### build up reducible r(n) iteratively #####
r(int n)=
{loc q=poly_null(rank);
if n%2==0 then
     loc k=n/2;
     for i=0 to k do
          q=q+1X[0,i,0,0,n-2*i];
     od;
fi;
if n%2==1 then
     loc k=(n-1)/2;
     for i=0 to n/2 do
          q=q+1X[0,i,0,0,n-2*i];
     od;
fi;
q;
}
```

respectively. Since the code calculates possible cohomologies by subtraction of multiplicities from left and right as described above, it will produce a certain over-counting of mutually excluding possibilities. For $D = 10$, this type of ambiguity is not present.

B. The content of $\mathcal{N}=1,\ D=10$ superfields

The "vertical" structure of an $\mathcal{N}=(1,0)$ scalar superfield in $D=10$ is as follows [22]. Each row contains the irreducible content of the completely antisymmetric product of ℓ chiral spinors $\wedge^{\ell}(00010)$.

$$
\begin{array}{rcccccc}
\ell=0 & (00000) & & & & \\
 & \downarrow & & & & \\
1 & (00010) & & & & \\
 & \downarrow & & & & \\
2 & (00100) & & & & \\
 & \downarrow & & & & \\
3 & (01001) & & & & \\
 & \downarrow & \searrow & & & \\
4 & (10002) & & (02000) & & \\
 & \downarrow & \searrow & \downarrow & & \\
5 & (00003) & & (11001) & & \\
 & \downarrow & \swarrow & \downarrow & & \\
6 & (01002) & & (20100) & & \\
 & \downarrow & \swarrow & \downarrow & & \\
7 & (10101) & & (30010) & & \\
 & \downarrow & \searrow & \downarrow & \searrow & \\
8 & (00200) & & (20011) & & (40000) \\
 & \downarrow & \swarrow & \downarrow & \swarrow & \\
9 & (10110) & & (30001) & & \\
 & \downarrow & \searrow & \downarrow & & \\
10 & (01020) & & (20100) & & \\
 & \downarrow & \searrow & \downarrow & & \\
11 & (00030) & & (11010) & & \\
 & \downarrow & \swarrow & \downarrow & & \\
12 & (10020) & & (02000) & & \\
 & \downarrow & \swarrow & & & \\
13 & (01010) & & & & \\
 & \downarrow & & & & \\
14 & (00100) & & & & \\
 & \downarrow & & & & \\
15 & (00001) & & & & \\
 & \downarrow & & & & \\
16 & (00000) & & & &
\end{array}
$$

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